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Probability Measures with Reflection Coefficients $\{a_n\} \in \ell^4$ and $\{a_{n+1} - a_n\} \in \ell^2$ are Erdős Measures

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In this article, we consider the system of polynomials orthogonal on the unit circle. It is proved that the conditions $\{a_n\} \in \ell^4$ and $\{a_{n+1} - a_n\} \in \ell^2$ imposed on the reflection parameters $\{a_n\}$ are sufficient for the associated measure to be Erdős one. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Consider polynomials $\varphi_n(z) = \alpha_n z^n + \cdots,\ \alpha_n > 0$ orthogonal on the unit circle with respect to some measure $\sigma(\theta)$ which has infinitely many growth points [2]. This means

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(z)\overline{\varphi_m(z)}\, d\sigma(\theta) = \delta_{nm},$$

where $z = e^{i\theta}$. Define $\Phi_n(z) = \frac{\varphi_n(z)}{\alpha_n}$. Then $\Phi_n(z)$ satisfies the system of difference equations

$$\begin{cases} \Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n\Phi_n^*(z), & \Phi_0(z) = 1, \\ \Phi_{n+1}^*(z) = \Phi_n^*(z) - a_n z\Phi_n(z), & \Phi_0^*(z) = 1, \end{cases} \tag{1}$$

where the reverse polynomials are defined as

$$\Phi_n^*(z) = z^n\bar{\Phi}(z^{-1}). \tag{2}$$

The coefficients a_n are called reflection (or Geronimus) parameters. It is not difficult to show that $|a_n| < 1$ for all $n = 0, 1, \ldots$ (see [2]). Consider any

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measure $\mu(\theta)$ on $[-\pi, \pi]$. Assume that it has non-trivial absolutely continuous component $\mu_{\mathrm{ac}}(\theta)$. We say that the essential support of μ_{ac} is the whole interval $[-\pi, \pi]$ if for any measurable subset C with positive Lebesgue measure ($|C| > 0$), we have the inequality $\mu(C) > 0$. Measures that satisfy this condition are often called the Erdős measures [5]. In [2], the following theorem was proved. We give its short version here.

THEOREM 1.1 (Geronimus [2]). *The following statements are equivalent*:

(a) *The function* $\ln \sigma'$ *is integrable. That is*

$$\int_{-\pi}^{\pi} \ln \sigma'(\theta)\, d\theta > -\infty. \tag{3}$$

(b) *The series* $\sum_{n=0}^{\infty} |a_n|^2$ *converges.*

(c) *There exists a subsequence* $\varphi_{n_\nu}^*(z)$ *bounded at least at one point inside the unit disk.*

Remark. It was also proved that if one of the conditions above holds, then the limit $\lim_{n\to\infty} \varphi_n^*(z) = \pi(z)$ exists. The convergence is uniform inside any disk $|z| \leqslant r < 1$. Function $\pi(z)$ is analytic and has no zeroes inside the unit disk. What is more, the following representation holds (formula (2.5) in [2]):

$$\pi(z) = \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln \sigma'(\theta)\, d\theta\right\}, \qquad |z| < 1. \tag{4}$$

Assume that we are given a sequence of coefficients a_n. The following theorem is true.

THEOREM 1.2 (Geronimus [2]). *An arbitrary choice of parameters* a_n *subjected to the single condition*

$$|a_n| < 1 \qquad (n = 0, 1, \ldots)$$

determines the entire orthogonal set $\Phi_n(z)$ *and the non-decreasing bounded function* $\sigma(\theta)$ *with infinitely many growth points.*

We are interested in the following question. What properties of $\{a_n\}$ provide the inclusion of σ in the Erdős class? It follows from Theorem 1.1 that σ is an Erdős measure, if $\{a_n\} \in \ell^2$ and $|a_n| < 1, = 0, 1, \ldots$. Similar results for different classes of parameters $\{a_n\}$ were obtained in [6, 7]. The subordinacy theory of [3] can also be effectively applied to this problem. In the present paper, we use a new approach. For a class of sequences $\{a_n\}$, we first establish an asymptotic formula for the polynomials $\varphi_n^*(z)$ for z from some real segment centered at zero. Then, we show that the asymptotics obtained guarantees the inclusion of σ into the Erdős class.

2. ASYMPTOTICS OF $\varphi_n^*(z)$

In this paragraph, we obtain asymptotics of $\varphi_n^*(z)$ and $\Phi_n^*(z)$. Note that [2, Formula (8.6)]

$$\frac{\alpha_0^2}{\alpha_n^2} = \prod_{k=0}^{n-1} \{1 - |a_k|^2\}. \tag{5}$$

Since $\{a_n\} \in \ell^4$, this implies that

$$\alpha_n = c_n \exp\left\{\frac{1}{2}\sum_{k=0}^{n-1} |a_k|^2\right\},$$

where $\{c_n\}$ is a convergent sequence. Consequently, the asymptotics for $\Phi_n^*(z)$ gives the asymptotic formula for

$$\varphi_n^*(z) = \alpha_n \Phi_n^*(z). \tag{6}$$

If $z = 0$, then the solution to the second equation of (1) is trivial, i.e., $\Phi_n^*(0) = 1,\ n = 0, 1, \ldots$, and we obtain the asymptotics for $\varphi_n^*(0)$ from (6). In what follows, we assume therefore that $z \neq 0$. Before stating the main result, we prove some auxiliary statements.

LEMMA 2.1. *Let* $\lim_{n\to\infty} a_n = 0$ *and* $\{a_{n+1} - a_n\} \in \ell^2$. *Then, the series*

$$\sum_{j=0}^{\infty} [\Re(\bar{a}_j a_{j+1}) - |a_j|^2] \tag{7}$$

converges.

Proof. Observing that

$$|a_{j+1} - a_j|^2 = |a_{j+1}|^2 + |a_j|^2 - 2\Re(\bar{a}_j a_{j+1})$$

and taking the sum of these identities, we obtain

$$\sum_{j=0}^{n} |a_{j+1} - a_j|^2 = \sum_{j=0}^{n} |a_{j+1}|^2 + |a_j|^2 - 2\Re(\bar{a}_j a_{j+1}).$$

It follows that

$$2\sum_{j=0}^{n}[\Re(\bar{a}_j a_{j+1})-|a_j|^2]=|a_{n+1}|^2-|a_0|^2-\sum_{j=0}^{n}|a_{j+1}-a_j|^2. \qquad \blacksquare$$

LEMMA 2.2. *Let $\{\Phi_n(z)\}$ be the sequence of the monic orthogonal polynomials corresponding to a sequence of the Geronimus parameters $\{a_n\}_{n\geqslant 0}$. Then,*

$$\Phi_n^*(z)=1-z\sum_{j=0}^{n-1}a_j\Phi_j(z), \tag{8}$$

$$\Phi_n(z)=z^n-z^{n-1}\sum_{j=0}^{n-1}\bar{a}_j z^{-j}\Phi_j^*(z), \tag{9}$$

$$\Phi_n^*(z)=1-\sum_{j=0}^{n-1}a_j z^{j+1}+\sum_{j=1}^{n-1}a_j z^j\sum_{l=0}^{j-1}\bar{a}_l z^{-l}\Phi_l^*(z), \tag{10}$$

$$\Phi_n^*(z)=1-\sum_{j=0}^{n-1}a_j z^{j+1}+\sum_{j=0}^{n-2}\Phi_j^*(z)\sum_{l=j+1}^{n-1}\bar{a}_j a_l z^{l-j}, \qquad n=1,2,\dots . \tag{11}$$

Proof. Formula (8) follows from the second recurrence of (1):

$$\begin{aligned}\Phi_n^*(z)&=\Phi_0^*(z)+\Phi_1^*(z)-\Phi_0^*(z)+\Phi_2^*(z)-\Phi_1^*(z)+\cdots+\Phi_n^*(z)-\Phi_{n-1}^*(z)\\&=1-z\sum_{k=0}^{n-1}a_k\Phi_k(z).\end{aligned}$$

Now (9) can be obtained from (8) by the $*$-operation defined in (2). Finally, (10) follows from (8) if we substitute (9) into it. Formula (11) is obtained from (10) by the change of the order of summation. $\blacksquare$

Denote

$$r_n=1-\sum_{j=0}^{n-1}a_j z^{j+1},$$

$$g_n=\bar{a}_n z^{-n}\sum_{j=n+1}^{\infty}a_j z^j,$$

$$l_n=-g_{n-1}\Phi_{n-1}^*(z)-\left(\sum_{l=0}^{n-2}\bar{a}_l z^{-l}\Phi_l^*(z)\right)\left(\sum_{j=n}^{\infty}a_j z^j\right).$$

In the next lemma, we obtain some useful formulas for $\Phi_n^*(z)$.

LEMMA 2.3. *The following relations for* $\Phi_n^*(z)$ *holds*:

$$\Phi_n^*(z) = r_n + l_n + \sum_{l=0}^{n-1} g_l \Phi_l^*(z), \tag{12}$$

$$\Phi_{n+1}^* - \Phi_n^* = r_{n+1} - r_n + l_{n+1} - l_n + g_n \Phi_n^*. \tag{13}$$

Proof. Equation (12) follows from (11) if we use notations r_n, l_n, and g_n. Equation (13) is the direct consequence of (12). ∎

DEFINITION. For a sequence of complex numbers, we define

$$\|\{x_n\}\|_\infty = \sup_{n \geqslant 0} |x_n|.$$

More generally, for any real positive p, we denote

$$\|\{x_n\}\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p}.$$

The following two lemmas establish some properties of the sequence $\{g_n\}$.

LEMMA 2.4. *The estimates*

$$\|\{g_n\}\|_\infty \leqslant \|\{a_n\}\|_\infty^2 \frac{|z|}{1-|z|}, \tag{14}$$

$$\|\{g_n\}\|_2 \leqslant \|\{a_n\}\|_4^2 \frac{|z|}{1-|z|} \tag{15}$$

hold.

Proof. Inequality (14) easily follows from the definition of g_n. To prove (15), use Cauchy's inequality first,

$$\|\{g_n\}\|_2 \leqslant \|\{a_n\}\|_4 \left\| \left\{ \sum_{j=n+1}^{\infty} a_j z^{j-n} \right\} \right\|_4.$$

The discrete version of Young's inequality for the convolution [4, pp. 239–240] states

$$\left\|\left\{\sum_{j=-\infty}^{\infty} x_j y_{n-j}\right\}\right\|_p \leqslant \|\{x_n\}\|_p \|\{y_n\}\|_1, \qquad p \geqslant 1. \tag{16}$$

Applying Young's inequality to

$$x_j = \begin{cases} a_j & \text{if } j \geqslant 0, \\ 0 & \text{if } j < 0. \end{cases} \quad \text{and} \quad y_j = \begin{cases} z^{-j} & \text{if } j < 0, \\ 0 & \text{if } j \geqslant 0 \end{cases}$$

with $p = 4$, we obtain that

$$\left\|\left\{\sum_{j=n+1}^{\infty} a_j z^{j-n}\right\}\right\|_4 \leqslant \|\{a_n\}\|_4 \|\{z^n\}_{n \geqslant 1}\|_1 = \frac{|z|}{1-|z|} \|\{a_n\}\|_4. \qquad \blacksquare$$

We denote by C a positive constant whose value may change from one formula to another.

Lemma 2.5. *The following asymptotics is true*:

$$\sum_{j=0}^{n} \Re g_j = \frac{z}{1-z} \sum_{j=0}^{n} |a_j|^2 + \omega_n(z), \tag{17}$$

where $\omega_n(z)$ tends to some $\omega_\infty(z)$ uniformly for real $z = x$ from some punctured vicinity of zero.

Proof. The trivial identity $x^k = (x^{k+1} - x^k)/(x-1)$ and the definition of g_n imply

$$\sum_{j=0}^{n} g_j = \sum_{j=0}^{n} \bar{a}_j x^{-j} \left(\sum_{k=j+1}^{\infty} a_k \frac{x^{k+1} - x^k}{x-1} \right).$$

Using the Abel transform, we have

$$\sum_{j=0}^{n} g_j = \frac{x}{1-x} \sum_{j=0}^{n} \bar{a}_j a_{j+1} + \frac{1}{x-1} \sum_{j=0}^{n} \bar{a}_j x^{-j} \sum_{k=j+2}^{\infty} (a_{k-1} - a_k) x^k.$$

Since $\{a_n\} \in \ell^4$ implies that $\lim_{n\to\infty} a_n = 0$, we obtain by Lemma 2.1

that

$$\frac{x}{1-x}\sum_{j=0}^{n}\Re\left(\bar{a}_j a_{j+1}\right)$$
$$=\frac{x}{1-x}\sum_{j=0}^{n}|a_j|^2+\frac{x}{1-x}\sum_{j=0}^{n}[\Re(\bar{a}_j a_{j+1})-|a_j|^2]$$
$$=\frac{x}{1-x}\sum_{j=0}^{n}|a_j|^2-\frac{x}{2(1-x)}\left(|a_0|^2+\sum_{j=0}^{\infty}|a_{j+1}-a_j|^2\right)+\frac{x}{1-x}\bar{o}(1).$$

To handle the second term, we use the elementary identity $x^{-j}=(x^{-j+1}-x^{-j})/(x-1)$. Now, we write

$$\sum_{j=0}^{n}\bar{a}_j\frac{x^{-(j-1)}-x^{-j}}{x-1}\sum_{k=j+2}^{\infty}(a_{k-1}-a_k)x^k$$
$$=\sum_{j=0}^{n}\bar{a}_j\frac{x^{-(j-1)}}{x-1}\sum_{k=j+2}^{\infty}(a_{k-1}-a_k)x^k-\sum_{j=1}^{n+1}\bar{a}_{j-1}\frac{x^{-(j-1)}}{x-1}\sum_{k=j+1}^{\infty}(a_{k-1}-a_k)x^k$$
$$=\frac{1}{x-1}\sum_{j=1}^{n}x^{-(j-1)}\left[\bar{a}_j\sum_{k=j+2}^{\infty}(a_{k-1}-a_k)x^k-\bar{a}_{j-1}\sum_{k=j+1}^{\infty}(a_{k-1}-a_k)x^k\right]$$
$$+\bar{a}_0\frac{x}{x-1}\sum_{k=2}^{\infty}(a_{k-1}-a_k)x^k-\bar{a}_n\frac{x^{-n}}{x-1}\sum_{k=n+2}^{\infty}(a_{k-1}-a_k)x^k. \tag{18}$$

The last term of (18) tends to zero uniformly on compact subsets of $(-1,1)$, since $\lim_{n\to\infty}a_n=0$ and since the Taylor coefficients of the power series in this term are in ℓ^4. Similarly, the Taylor series of the second term in (18) converges uniformly on the compact subsets of $(-1,1)$. The first term in (18) can be written as follows:

$$\frac{x}{x-1}\sum_{j=1}^{n}x^{-j}(\bar{a}_j-\bar{a}_{j-1})\sum_{k=j+2}^{\infty}(a_{k-1}-a_k)x^k-\frac{x^2}{x-1}\sum_{j=1}^{n}\bar{a}_{j-1}(a_j-a_{j+1}). \tag{19}$$

Apply Cauchy's inequality

$$\sum_{j=1}^{n}|x_j|\,|y_j|\leqslant\left(\sum_{j=1}^{n}|x_j|^2\right)^{1/2}\left(\sum_{j=1}^{n}|y_j|^2\right)^{1/2},$$

with $x_j=\bar{a}_j-\bar{a}_{j-1}$ and $y_j=x^{-j}\sum_{k=j+2}^{\infty}(a_{k-1}-a_k)x^k$, to the first term in (19). $x_j\in\ell^2$ and $y_j\in\ell^2$ by discrete Young's inequality. Therefore, the first term of (19) converges uniformly near the origin. For the second one, we have

$$\sum_{j=1}^{n}(a_j-a_{j+1})(\bar{a}_{j-1}-\bar{a}_j+\bar{a}_j)=\sum_{j=1}^{n}(a_j-a_{j+1})(\bar{a}_{j-1}-\bar{a}_j)+\sum_{j=1}^{n}(a_j-a_{j+1})\bar{a}_j.$$

The first series converges by Cauchy's inequality. The real part of the second one converges by Lemma 2.1. ∎

Consider

$$S_n=\prod_{j=0}^{n-1}(g_j+1) \tag{20}$$

for real $z=x$ from the small punctured vicinity of zero. S_n is the solution of the equation $S_{n+1}-S_n=S_n g_n$, $S_0=1$. The next lemma establishes the asymptotics for $|S_n|$.

Lemma 2.6. *$|S_n|$ admits the following asymptotics*:

$$|S_n|=\exp\left(\frac{x}{1-x}\sum_{j=0}^{n}|a_j|^2\right)v_n(x), \tag{21}$$

where $v_n(x)$ tends to some positive $v_\infty(x)$.

Proof. From (20), we infer

$$|S_n|=\exp\left\{\Re\sum_{j=0}^{n-1}\ln(1+g_j)\right\}.$$

Use the Taylor expansion for logarithm: $\ln(1+s)=s+O(s^2)$ as $s\to 0$. Lemma 2.5 yields the asymptotics for $\sum_{j=0}^{n-1}\Re g_j$. Boundedness of $\sum_{j=0}^{n-1}|g_j|^2$ follows from Lemma 2.4. ∎

For Φ_n^*, we have Eq. (12). Let us find Φ_n^* in the following form $\Phi_n^*=S_nD_n$. Then, the study of the sequence D_n is reduced to the analysis of the equation from the following lemma.

Lemma 2.7. *We have the following equation for D_n*:

$$D_n=D_2+\sum_{j=2}^{n-1}\frac{r_{j+1}-r_j}{S_{j+1}}+\frac{l_n}{S_n}-\frac{l_2}{S_3}+\sum_{j=3}^{n-1}l_j\left(\frac{1}{S_j}-\frac{1}{S_{j+1}}\right)\qquad (n=2,3,\ldots), \tag{22}$$

where the formula for l_n *is*

$$l_n = -g_{n-1}S_{n-1}D_{n-1} - \left(\sum_{l=0}^{n-2} \bar{a}_l x^{-l} S_l D_l\right)\left(\sum_{j=n}^{\infty} a_j x^j\right). \tag{23}$$

Proof. Substituting the factorization $\Phi_n^* = S_n D_n$ in (13) and using (20), we obtain

$$D_{n+1} = D_n + \frac{r_{n+1} - r_n + l_{n+1} - l_n}{S_{n+1}}.$$

Summing up these equations, we obtain (22) by the Abel transform. ∎

Now we are in the position to prove the main result of this section.

THEOREM 2.1. *If the reflection parameters* a_n *are such that* $|a_n| < 1$, $a_n \in \ell^4$, *and* $a_{n+1} - a_n \in \ell^2$, *then we have the asymptotics*

$$|\varphi_n^*(z)| = \exp\left\{\frac{1+z}{2(1-z)} \sum_{j=0}^{n-1} |a_j|^2\right\} k_n(z), \tag{24}$$

where $k_n(z)$ *tends to some positive* $k_\infty(z)$ *uniformly for real* $z = x$ *in some small neighborhood of zero* $|x| < \delta$.

Proof. Let us establish the following asymptotics for $\Phi_n^*(x)$. If $|x| < \delta$, then

$$|\Phi_n^*(x)| = \exp\left\{\frac{x}{1-x} \sum_{j=0}^{n-1} |a_j|^2\right\} w_n(x), \tag{25}$$

where $w_n(x)$ tends to some positive $w_\infty(x)$ uniformly for real $z = x$, $(|x| < \delta)$, and δ is some small number that depends on $\|\{a_n\}\|_4$ and $\|\{a_{n+1} - a_n\}\|_2$. Once this formula is obtained, we can use (6) to prove (24). Consider Eq. (22). Subtract unity from the both sides of (22). Let us show that

$$\max_{j=0,\ldots,n} |D_j - 1| \leqslant T_1(x) + T_2(x) \max_{j=0,\ldots,n} |D_j - 1|, \tag{26}$$

where $T_1(x), T_2(x) \to 0$ as $x \to 0$. Indeed, estimate terms on the right-hand side of (22). From Lemma 2.6, we have $|S_{j+1}| > C > 0$ for $|x|$ small enough. Therefore,

$$\left|\sum_{j=2}^{n-1} \frac{r_{j+1} - r_j}{S_{j+1}}\right| \leqslant C \sum_{j=2}^{n-1} |a_j|\, |x|^{j+1} \leqslant C|x|. \tag{27}$$

For l_2/S_3,

$$\left|\frac{l_2}{S_3}\right| \leqslant C|g_1|\,|\Phi_1^*| + C|a_0|\,|\Phi_0^*| \sum_{j=2}^{\infty} |a_j|\,|x|^j \leqslant C|x|, \tag{28}$$

because

$$|g_1| \leqslant |a_1|\,|x|^{-1} \sum_{j=2}^{\infty} |a_j|\,|x|^j \leqslant C|x|.$$

The estimate on l_n/S_n from the right-hand side of (22) is

$$\begin{aligned}\left|\frac{l_n}{S_n}\right| &\leqslant C|g_{n-1}|\,|D_{n-1}| \\ &\quad + C\left(1 + \max_{l=0,\dots,n-2} \{|D_l - 1|\}\right)|S_n|^{-1}\left(\sum_{l=0}^{n-2} |a_l|\,|x|^{-l}|S_l|\right)\left(\sum_{j=n}^{\infty} |a_j|\,|x|^j\right) \\ &\leqslant C|x| \max_{j\geqslant n} \{|a_j|\}\left(1 + \max_{l=0,\dots,n-2}\{|D_l - 1|\}\right).\end{aligned} \tag{29}$$

The difference in the last term from the right-hand side of (22) can be written as follows:

$$\frac{1}{S_j} - \frac{1}{S_{j+1}} = \frac{g_j}{S_{j+1}}.$$

Therefore, using formula (23) for l_j, we get

$$\left|\sum_{j=3}^{n-1} l_j\left(\frac{1}{S_j} - \frac{1}{S_{j+1}}\right)\right| \leqslant A_1 + A_2, \tag{30}$$

where

$$\begin{aligned}A_1 &= C\left(1 + \max_{j=2,\dots,n-2} \{|D_{j-1} - 1|\}\right)\sum_{j=3}^{n-1} |g_{j-1}|\,|S_{j-1}|\frac{|g_j|}{|S_{j+1}|} \\ &\leqslant C\left(1 + \max_{j=2,\dots,n-2} \{|D_{j-1} - 1|\}\right)\|\{g_n\}\|_2^2 \\ &\leqslant C|x|^2\left(1 + \max_{j=0,\dots,n} \{|D_j - 1|\}\right).\end{aligned} \tag{31}$$

The last inequality follows from (15). For A_2, we have

$$A_2 = \left| \sum_{j=3}^{n-1} \left(\sum_{l=0}^{j-2} \bar{a}_l x^{-l} S_l D_l \right) \left(\sum_{k=j}^{\infty} a_k x^k \right) \frac{g_j}{S_{j+1}} \right|$$
$$\leqslant C \left(1 + \max_{j=0,\ldots,n-3} \{|D_j - 1|\} \right) \sum_{j=3}^{n-1} \left[\left(\sum_{l=0}^{j-2} |a_l|\,|x|^{-l} \right) \left(\sum_{k=j}^{\infty} |a_k|\,|x|^k \right) |g_j| \right].$$

Here we used the asymptotics of S_n established in Lemma 2.6. Apply the generalized Cauchy's inequality

$$\sum_{j=3}^{\infty} |x_j|\,|y_j|\,|z_j| \leqslant \|\{x_j\}_{j\geqslant 3}\|_2 \|\{y_j\}_{j\geqslant 3}\|_4 \|\{z_j\}_{j\geqslant 3}\|_4,$$

with $x_j = g_j,\ \ y_j = x^j \sum_{l=0}^{j-2} |a_l|\,|x|^{-l},\ \ z_j = x^{-j} \sum_{k=j}^{\infty} |a_k|\,|x|^k$. Now it suffices to use discrete Young's inequality for $\{y_j\}$, $\{z_j\}$, and (15) for $\{x_j\}$ to obtain the estimate

$$A_2 \leqslant C|x| \left(1 + \max_{j=0,\ldots,n} \{|D_j - 1|\} \right). \tag{32}$$

Note that $\lim_{x\to 0} D_2(x) = 1$. Let

$$T_1(x) = |D_2 - 1| + \left| \sum_{j=2}^{n-1} \frac{r_{j+1} - r_j}{S_{j+1}} \right| + \left| \frac{l_2}{S_3} \right| + C|x| \max_{j\geqslant n} |a_j| + C|x|$$

and

$$T_2(x) = C|x| \max_{j\geqslant n} |a_j| + C|x|.$$

From (27)–(32), we infer (26) with $\lim_{x\to 0} T_1(x) = 0$, $\lim_{x\to 0} T_2(x) = 0$. The following estimate holds:

$$\max_{j=0,\ldots,n} |D_j - 1| \leqslant \frac{T_1(x)}{1 - T_2(x)}. \tag{33}$$

Because $T_1(x), T_2(x) \to 0$ as $x \to 0$, it implies that $|D_n|$ is bounded above and below from zero for $|x|$ sufficiently small. Due to (29) and boundedness of D_n, l_n/S_n converges to zero uniformly for x from some punctured vicinity of origin. Then, we apply Cauchy's criterion of uniform convergence to series from (22). Estimates on S_n, boundedness of D_n, and inequalities analogous to (27), (30)–(32) yield that the series in (22) converge absolutely and uniformly in the neighborhood of zero. Therefore, $D_n(x)$ converges to some $D(x)$. This convergence is uniform for x from some punctured vicinity of zero. $|D(x)|$ is bounded above and below from zero. Due to factorization $\Phi_n^*(z) = S_n D_n$ and Lemma 2.6, we have asymptotics (25) and consequently (24). ∎

Remark. Because $D_2(z)$ is analytic in $z = 0$ and $D_2(0) = 1$, (22) and (27)–(32) yield the inequalities: $0 \leqslant T_1(x) \leqslant C|x|$, $0 \leqslant T_2(x) \leqslant C|x|$.

3. THE PRESENCE OF A.C. COMPONENT

The main result of the article is

THEOREM 3.1. *Under the conditions of Theorem* 2.1, *the associated measure* $\sigma(\theta)$ *has absolutely continuous component*, *whose essential support is* $[-\pi, \pi]$.

Proof. For the systems with $a_j^{(n)} = \begin{cases} a_j, & j \leqslant n, \\ 0, & j > n \end{cases}$, we have formula (4)

$$\pi^{(n)}(z) = \exp\left\{ -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \ln \sigma_n'(\theta)\, d\theta \right\}, \tag{34}$$

where $\pi^{(n)}(z) = \lim_{k\to\infty} \varphi_k^{(n)*}(z) = \varphi_n^*(z)$. Choosing $z = 0$ and then $z = \gamma$ $(0 < \gamma < \delta)$, we have

$$\frac{2\gamma(1+\gamma)}{4\pi(1-\gamma)} \int_{-\pi}^{\pi} \frac{1 - \cos\theta}{1 + \gamma^2 - 2\gamma\cos\theta} \ln \sigma_n'(\theta)\, d\theta = |\varphi_n^*(\gamma)\varphi_n^*(0)^{(1+\gamma)/(\gamma-1)}|, \tag{35}$$

where the right-hand side tends to some positive constant due to (24). Thus, we obtain the uniform boundedness of $\int_{-\pi}^{\pi} \frac{1-\cos\theta}{1+\gamma^2-2\gamma\cos\theta} \ln \sigma_n'(\theta)\, d\theta$ in n. Now it suffices to use one argument that was applied to Sturm–Liouville operators by Deift and Killip [1]. The measure $\sigma_n(\theta)$ converges weakly to $\sigma(\theta)$. We have the trivial inequality $\ln^+ t < t$, where $\ln^+ t = \ln t$ if $t > 1$ and equals to zero if $0 < t < 1$. Because $\int_{-\pi}^{\pi} \sigma_n'(\theta)\, d\theta$ is bounded in n, $\int_{[a,b]} \ln^+ \sigma_n'(\theta)\, d\theta$ is bounded as well, where $[a, b]$—any segment that does not contain zero. Therefore, $\int_{[a,b]} \ln^- \sigma_n'(\theta)\, d\theta$ is bounded uniformly in n, where $\ln^- t = -\ln t$ for $0 < t < 1$ and is equal to zero for $t > 1$. Given any compact C, such that $|C| > 0$, $dist(0, C) > 0$, use Jensen's inequality

$$\ln^- \left\{ \frac{1}{|C|} \int_C \sigma_n'(\theta)\, d\theta \right\} \leqslant \frac{1}{|C|} \int_C \ln^- \sigma_n'(\theta)\, d\theta \tag{36}$$

to prove that $\sigma_n(C) \geqslant d(C) > 0$ for any n. Because σ_n converges weakly to σ, $\sigma(C) \geqslant \limsup_{n\to\infty} \sigma_n(C) > 0$. If C is such that $dist(0, C) = 0$ and $|C| > 0$, we can always find compact subset C_1 that satisfies $dist(0, C_1) > 0$ and $|C_1| > 0$. For this subset C_1, we can use the same argument. ∎

Remark. The rotation of the circle to the angle τ changes the reflection coefficients as follows $\hat{a}_n = e^{-i(n+1)\tau} a_n$ (it follows, for example, from the continued fraction expansion of the associated Schur function, see [5]). Therefore, we can change condition $a_{n+1} - a_n \in \ell^2$ in Theorem 3.1 to $e^{i\tau} a_{n+1} - a_n \in \ell^2$ for some real τ.

Remark. It is likely that condition $a_n \in \ell^4$ in Theorem 3.1 can be relaxed to $a_n \to 0$.

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